Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 636–646
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Bifurcations of a plane crack front growing quasistatically in an elastic space☆

S.A. Nazarov

St. Petersburg, Russia

Received 29 September 2005

Abstract

Using variational-asymptotic models of force and energy criteria, situations are found in which bifurcations of the form of the front accompanying the quasistatic propagation of a plane crack in an elastic isotropic space are possible. Two types of bifurcations are revealed for a circular crack in the case of axisymmetric loading: fluctuation of the centre of the crack while preserving its circular form and distortion of the front due to the formation of two or a larger number of "lobes".

1. The variational-asymptotic model of the quasistatic development of a crack

Consider a crack M bounded by a simple smooth closed contour Γ and located in the plane $\{x \in \mathbb{R}^3 : z{:=}x_3 = 0\}$ in an isotropic elastic space with Lamé constants $\lambda \geq 0$ and $\mu > 0$. The crack is opened by symmetrical normal forces applied to its edges

$$g^{\pm}(\tau; y) = \mp g(y) \mp \tau g'(y), \quad y = (y_1, y_2) \in M^{\pm} \tag{1.1}$$

Here τ is a time-like parameter. If we assume that the initial ($\tau = 0$) load $\mp g$ is the critical load, its variation according to representation (1.1) can lead to the crack growth. We will denote the position of the crack at the instant $\tau > 0$ by $M(\tau)$ and the front by $\Gamma(\tau)$. Assuming that the fracture process is irreversible and that the crack remains plane, we obtain the inclusion $M(\tau_1) \subset M(\tau_2)$ when $\tau_2 \geq \tau_1$ and the following description of the contour $\Gamma(\tau)$ as a perturbation for a small $\tau > 0$ of the initial contour Γ

$$\Gamma(\tau) = \{y \in \mathbb{R}^2 : n = h(\tau; s)\}, \quad h \geq 0 \tag{1.2}$$

Here, (n, s) are natural curvilinear coordinates in the neighbourhood of the set Γ, s is the length of an arc and n is the distance in the plane along the outward normal, $n > 0$ outside M. The function $h(\tau; \cdot)$ describes the position of the crack front at the instant τ, and, in order to ensure a quasistatic process in the segment $[0, \tau_0]$, it is necessary that the function $h(\cdot; s)$ should be uniformly continuous with respect to s in this segment and infinitesimal when $\tau \to +0$. The possibility of ignoring the inertial terms is ensured by the following supposition: a change in the dimensionless loading parameter

☆ *Prikl. Mat Mekh.* Vol. 70, No. 4, pp. 701–713, 2006.
E-mail address: serna@snark.ipme.ru.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2006.09.014

τ occurs slowly in real time, which can be characterized, for example, by the ratio $c^{-1}\mathrm{diam}M$, where c is the velocity of elastic waves.

The variational nature of the fracture process, which is revealed in a two-dimensional situation[1–3] enables us to devise[4–6] variational-asymptotic models of three-dimensional criteria of the Irwin (force) and Griffith (energy) fracture which were subsequently developed in Refs. 7–9. Since the condition $h \geq 0$ in formula (1.2) requires the formulation of linear constraints in the problem of determining h, the above-mentioned models are described by variational inequalities of the form

$$(bH, X-H)_\Gamma - (B(H), X-H)_\Gamma \geq (F, X-H)_\Gamma \tag{1.3}$$

Here, $(\ ,\ )_\Gamma$ is a scalar product in the space $L_2(\Gamma)$ and X is an arbitrary test function which is smooth and non-negative. The function h in formula (1.2) is recovered using the solution $H \geq 0$ of problem (1.3) (see (2.12) below). The data b, B and F in equality (1.3) are constructed from the solutions of elastic problems for a fixed crack M, that is, when $\tau = 0$ (see Section 2).

A complete investigation of problem (1.3) enables us to answer the question of the character of the crack propagation for a narrow range of variation of the parameter τ. For instance, if, when $\tau \leq \tau_0$, the variational inequality (1.3) has a unique solution and the solution turns out to be small (zero), then, when $\tau \in [0, \tau_0]$, the crack develops *quasistatically and stably* (it remains *unchanged*). If, however, there are several small solutions, then one has to speak of *bifurcations of the form of the front* of the advancing crack. Finally, in the case when there are no solutions or there are only large-value solutions which do not disappear when $\tau \to +0$, the growth of the crack becomes of the *avalanche-type*.

We emphasize that the derivation of the inequality (1.3) is based on a formula for the stress intensity factor (SIF) $K_1(\tau; s)$ on the front (1.2) of the crack $M(\tau)$ in the case of the load (1.1) (see Refs. 4,10–13 etc.) but this asymptotic formula ceases to work for "large depths" $h(s)$. The absence of small solutions in the case of the non-linear problem (1.3) is therefore precisely linked to a cessation of the quasistatic fracture process and the need to take account of dynamic effects (the crack "jumps over" from one position to another). Moreover, in a study of the long quasistatic development of a crack, it is necessary, having subdivided the interval $(0, T]$ into fragments (τ_i, τ_{i+1}), to solve problem (1.3) iteratively: the solution $h_i(\tau - \tau_i; s)$ on the base contour Γ_i enables one to determine the next base contour $\Gamma_{i+1} = \Gamma_i(\tau_{i+1} - \tau_i)$, after which it is necessary to recalculate the data b_{i+1}, B_{i+1} and F_{i+1} for the new position of the front Γ_{i+1}.

Examples of the quasistatic and avalanche-type growth of a crack are well known. An asymptotic and numerical solution of problem (1.3) was obtained in Refs. 6,7,14 for canonical forms of cracks and specific loads. In particular, it was established that, in the case when the stress intensity factor reaches a critical value K_{1c} at just a single point s_0 on the contour Γ, the start of the motion of the crack is characterized by a non-smooth dependence of the magnitude of $h(\tau; s)$ on the parameter τ. The first derivative $\partial_\tau h(0; s)$ is equal to zero and the second does not exist ! This effect is explained by the different rates of propagation of a crack branch into the depth $O(\tau^{3/2})$ and into the width $O(\tau^{1/2})$. In this case, the asymptotic formula for h appears as

$$h(\tau; s) \sim \tau^{3/2}\mathbf{h}(\tau^{-1/2}(s - s_0)) \tag{1.4}$$

Here $\mathbf{h}$ is a certain reference function (see Refs. 6,14). The incorrect impression[15] that the Griffith criterion is not applicable to the determination of the form of a crack arose precisely because of the irregular behaviour of the front (1.2) at the initial stage of the development of a crack and the fact that classical perturbation theory cannot be used in conditions (1.4). Several investigations[16,17] led to the paradoxical conclusion that $\partial_\tau h(0; s) = h_0\delta(s - s_0)$. This form contains the Dirac δ-function and means that only one point s_0 starts off from the whole front Γ !?.

Only one publication, Ref. 18, which deals with the problem of the bifurcation of crack fronts in three-dimensional bodies is known to us (as regards two-dimensional problems, see Refs. 2,19 and others). A tunnel crack

$$M = \{x = (y, z): |y_1| < a,\ y_2 \in \mathbb{R},\ z = 0\} \tag{1.5}$$

was, in fact, considered in Ref. 18 and it was shown that, under a constant tensile stress at infinity σ_{zz}^∞, small perturbations

$$\{x = (y, z) \in \mathbb{R}^3 : y_1 = \pm a \pm h_\pm(y_2),\ y_2 \in \mathbb{R},\ z = 0\} \tag{1.6}$$

of the sides of the strip (1.5), in which the SIF factor $K_1^\pm(x_2)$ maintains a constant (critical) value, exist. The functions $h_\pm$ in formula (1.6) are found to be periodic with a period which depends on a but they do not satisfy the conditions

$h_\pm \geq 0$, and this means that it is difficult to interpret this mathematical fact within the framework of fracture mechanics. Moreover, according to results obtained earlier,[20,21] the question of the correctness of the formulation of the above-mentioned conditions at infinity requires a separate investigation, and the answer to it is not so obvious on account of the unboundedness of the surface (1.5).

In this paper, a number of examples of the general scheme and specific examples are presented in which the variational inequality (1.3) has a whole *family* of small positive solutions which enables us to speak of the bifurcations of a crack front accompanying quasistatic fracture in the cases of force and energy criteria. We emphasize that a *positive* solution H of problem (1.3) satisfies the equations

$$bH - B(H) = F \text{ on } \Gamma \tag{1.7}$$

Hence, in the formation of the variational inequality (1.3) on the basis of the Irwin criterion, the SIF $K_1(\tau; s)$ on the front (1.2) of the crack $M(\tau)$ is equal to $K_{1c} + O(\tau^2)$ and maintains a constant value within the limits of the accuracy of the variational-asymptotic model.

2. Derivation of the variational inequality

It is well known (see Ref. 22, for example) that, using the Papkovich–Neuber representation, an elastic problem for a half-space with a crack can be simplified to a mixed scalar boundary-value problem in the half-space $\mathbb{R}^3_+ = \{x : z > 0\}$ and, in fact,

$$-\Delta_x v(x) = 0, \quad x \in \mathbb{R}^3_+; \quad \partial_z v(y, 0) = -\alpha g(y), \quad y \in M; \quad v(y, 0) = 0, \quad y \in \mathbb{R}^2 \backslash \overline{M} \tag{2.1}$$

Here $\alpha = \mu^{-1}(1 - \nu)$, $\mu > 0$ is the shear modulus, $\nu = \lambda[2(\lambda + \mu)]^{-1}$ is Poisson's ratio and the function g in the crack M is taken from expansion (1.1). It is also known that a unique solution of problem (1.1) exists which has a finite Dirichlet integral $||\nabla_x v; L_2(\mathbb{R}^3_+)||^2$ and therefore vanishes at infinity. Apart from this solution v, a further solution v' is required which is obtained by replacing the quantity g in boundary condition (2.1) by the second term g' from expansion (1.1). Since the traces in the plane $\partial\mathbb{R}^3_+$ of the displacement u_z and stress $\sigma_{zz}(u)$ are identical with the functions $v(\cdot, 0)$ and $\alpha^{-1}\partial_z v(\cdot, 0)$ respectively, all the required characteristics of the stress-strain state around the crack front are calculated using the solutions of a scalar problem.

Assuming that g and g' are continuous functions in the set M right up to its boundary Γ, we will write the asymptotic representation of the solution $v \in C^\infty(\overline{\mathbb{R}^3_+} \backslash \Gamma)$ of problem (2.1) close to the edge Γ (see Refs. 4,8 and, also, Ref. 23) as

$$\begin{aligned} v(x) &= \alpha\left(\frac{2}{\pi}\right)^{-1/2}\left\{r^{1/2}K_1(s)\sin\frac{\varphi}{2} + \frac{1}{3}r^{3/2}k_1(s)\sin\frac{3\varphi}{2}\right\} - \alpha r g_\Gamma(s)\sin\varphi + \\ &+ \alpha\left(\frac{2}{\pi}\right)^{1/2} r^{3/2}\kappa(s)K_1(s)\left[\frac{1}{4}\sin\frac{\varphi}{2} - \frac{1}{12}\sin\frac{3\varphi}{2}\right] + O(r^2) \end{aligned} \tag{2.2}$$

Here $\kappa(s)$ is the curvature of the contour Γ at the point s, (r, φ) are the polar coordinates in planes perpendicular to Γ and the edges $M^\pm$ of the crack are specified locally by the equalities $\varphi = \pm\pi$. Moreover, $K_1(s)$ and $k_1(s)$ are the SIF of the first mode and the coefficient of the lowest singularity, determined using the normal tensile stresses on the crack extension ($\varphi = 0$) according to the asymptotic representation

$$\sigma_{zz}(u; y, 0) = \alpha^{-1}\partial_z v(y, 0) = (2\pi r)^{-1/2}\{K_1(s) + k_1(s)r\} - g_\Gamma(s) + O(r^2)$$

The contraction of the function g on the edge Γ is implied by g_Γ. The SIF, corresponding to the forces g' and the solution v' is denoted by $K_1{}'(s)$.

The asymptotic formula for the SIF $K_1(\tau; s)$ on the edge (1.2) for the load (1.1) plays a key role in constructing a variational-asymptotic model based on the force criterion of fracture. The point is that the a posteriori formulation of the Irwin criterion (for two-dimensional[2] and three-dimensional[4] bodies)

$$h(\tau; s) > 0 \Rightarrow K_1(\tau; s) = K_{1c}$$
$$h(\tau, s) = 0 \Rightarrow K_1(\tau; s) \leq K_{1c}$$

which contains the critical value of the SIF K_{1c}, is equivalent to the variational inequality

$$(K_1(\tau;\cdot) - K_{1c}, X - K_1(s)h(\tau;\cdot))_\Gamma \le 0, \quad \forall X \in C^\infty(\Gamma), \quad X \ge 0 \tag{2.3}$$

and substitution of the leading terms of the asymptotic K_1 into the left-hand side of inequality (2.3) leads to an inequality of the form (1.3). In the case of a following load (that is, the forces on the newly formed surface $M(\tau)\setminus\bar{M}$ have the previous form (1.1); compare with Ref. 24), the above formula, which was obtained under different conditions and using different methods (see Refs. 10–13,4 etc.), takes the form:

$$\begin{aligned} K_1(\tau;s) &\sim K_1(s) + \tau K_1'(s) + B(K_1h(\tau;\cdot);s) + \\ &+ \left\{Z^\#(s) - \frac{3}{8}\kappa(s) + \frac{1}{2}K_1(s)^{-1}k_1(s)\right\}K_1(s)h(\tau;s) \end{aligned} \tag{2.4}$$

In addition to the quantities K_1, k_1 and K'_1, taken from relations (2.2) for v and v', an integral operator B occurs in formula (2.4) with a positive symmetric kernel $Z(s, t)$ and a discontinuity $Z^\#(s)$ of the regular part of the primitive $\hat{Z}(s, t)$ of the function $\Gamma\setminus\{s\} \ni t \mapsto Z(s, t)$:

$$B(H;s) = \int_\Gamma (H(t) - H(s))Z(s,t)dt \tag{2.5}$$

$$\begin{aligned} &Z(s,t) = \frac{1}{2\pi}|s-t|^{-2} + O(1), \quad \hat{Z}(s,t) = \frac{1}{2\pi}(s-t)^{-1} + Z^0(s,t) \\ &Z^\#(s) = Z^0(s,s+0) - Z^0(s,s-0), \quad Z^0 \in C^\infty(\Gamma\setminus\{s\}) \end{aligned} \tag{2.6}$$

The integral operator (2.5) originates from the identity

$$V(H;x) = \int_\Gamma H(t)\zeta(t;x)dt, \quad \forall H \in C^1(\Gamma) \tag{2.7}$$

which relates the two types of weighting functions: the first, $V(H;\cdot)$, has a singularity $O(r^{-1/2})$ distributed along the whole of the edge Γ with a density H, and a stronger singularity $O(\rho_t^{-3/2})$ in the second function $\zeta(t;\cdot)$, which is concentrated just at one point $t \in \Gamma$. In other words, the following expansions hold

$$\begin{aligned} V(H;x) = \left(\frac{1}{2\pi}\right)^{1/2}&\left\{r^{-1/2}H(s)\sin\frac{\varphi}{2} - \kappa(s)r^{1/2}H(s)\left[\frac{1}{4}\sin\frac{3\varphi}{2} - \frac{3}{4}\sin\frac{\varphi}{2}\right] + \right. \\ &\left. + 2r^{1/2}\left(B(H;s) + Z^\#(s)H(s) + \frac{3}{8}\kappa(s)H(s)\right)\right\} + O(r^2) \end{aligned} \tag{2.8}$$

$$\zeta(t;x) = \left(\frac{1}{2\pi^3}\right)^{1/2}\rho_t^{-3/2}(\sin\theta_t)^{1/2}\sin\left(\frac{\varphi}{2}\right) + O(\rho_t^{-1/2}), \quad \rho_t \le \varepsilon \tag{2.9}$$

$$\zeta(t;x) = \left(\frac{2}{\pi}\right)^{1/2} r^{1/2}Z(t,s)\sin\frac{\varphi}{2} + O(r^{3/2}), \quad \rho_t \ge \varepsilon \tag{2.10}$$

Here, ε is an arbitrary positive number, and $(\rho_t, \theta_t, \varphi)$ are "spherical" coordinates corresponding to the coordinates $(r, \varphi, s-t)$, which are considered as "cylindrical" coordinates, that is, $\rho_t^2 = r^2 + (s-t)^2$ and $r = \rho_t\sin\theta_t$. Note that comparison of the representations (2.9) and (2.10), which hold in the ε-neighbourhood of the point $s \in \Gamma$ and on the remaining part of the edge respectively, leads to the first relation in (2.6).

The weighting functions V and ζ occur in the following integral formulae for the SIF from the asymptotic representation (2.2) of the solution v of problem (2.1)

$$\int_\Gamma H(s)K_1(s)ds = 2\int_M V(H;y,0)g(y)dy, \quad K_1(s) = 2\int_M \zeta(s;y,0)g(y)dy \tag{2.11}$$

The doubling of the integrals in (2.11) is due to the same loading on the two edges of the crack.

The occurrence of two types of weighting function was initiated in Refs. 25–27 and also Refs. 28,29 (see also, Refs. 4,8,30 etc.). The identity (2.7) was established for the first time in Ref. 4 and a general theory of weighting functions was constructed in Refs. 23,31

Multiplying relation (2.3) by $-K_{1c}^{-2}$ and replacing the SIF $K_1(\tau; s)$ by its asymptotic form (2.4), after some simple reduction we arrive at inequality (1.3), in which

$$H(\tau; s) = K_{1c}^{-1}K_1(s)h(\tau; s) \tag{2.12}$$

$$b(s) = \frac{3}{8}\kappa(s) - Z^{\#}(s) - \frac{1}{2}K_1(s)^{-1}k_1(s), \quad F(\tau; s) = K_{1c}^{-1}K_1(s) - 1 + \tau K_{1c}^{-1}K_1'(s) \tag{2.13}$$

We emphasize that, without a renormalization of the unknown (2.12), the left-hand side of inequality (1.3) loses the property of a symmetric quadratic form. The inequality $K_1(s) > 0$, which certainly holds in the case of the opening of a crack, ensures the consistency of the definitions (2.12) and (2.13): the quantity H inherits non-negativity from h and the quantity b remains bounded. It is only when $K_1(s) = K_{1c}$, that is, when $F(\tau; s) = \tau F_0(s)$ (the load $g^{\pm}(0; y)$ makes the crack M critical at all points of the front Γ) that the parameter τ is eliminated from problem (1.3) and its solution depends linearly on τ:

$$H(\tau; s) = h(\tau; s) = \tau h_0(s) \tag{2.14}$$

The variational-asymptotic model of the Griffith criterion, which requires that a minimum of the total energy (the sum of the potential and surface energies) $T(\tau)$, that has been accumulated by the space with a crack $M(\tau)$ in the case of a load (1.1) must be found, is obtained by replacing the functional $T(\tau)$ by its three-term asymptotic form (for details, see Refs. 5,7) and reduces to the variational inequality (1.3), where B and H are determined using formulae (2.5) and (2.12), but

$$\begin{aligned} b(s) &= \kappa(s)\left(\frac{7}{8} - \frac{1}{2}K_1(s)^{-2}K_{1c}^2\right) - Z^{\#}(s) - \frac{1}{2}K_1(s)^{-1}k_1(s) \\ F(\tau; s) &= \frac{1}{2}\left(\frac{K_{1c}}{K_1(s)} - \frac{K_1(s)}{K_{1c}}\right) + \frac{\tau}{K_{1c}}K_1'(s) \end{aligned} \tag{2.15}$$

Since expressions (2.15) and (2.13) differ only slightly from one another and, in the study of problem (1.3), the conditions are imposed directly on b and F, a further type of fracture criterion is not specified. Note that the inequality $H_I \geq H_G$ was established in Refs. 5 and 8 in the case of convex cracks, which means that the force criterion predicts a somewhat greater advance of the crack front than the energy criterion. This difference is mainly due to the curvature of the contour Γ (compare the first terms in (2.13) and (2.15)) and, in the two-dimensional case $H_I(\tau; s) - H_G(\tau; s)$, the difference lies[9] within the limits of the accuracy $O(\tau^2)$ of the models themselves.

The second of the representations (2.11) for $K_1(s)$ and the analogous integral formulae for $k_1(s)$ shows that any specified values can be assigned in advance to these coefficients by varying the load. This observation is used in the case of a circular crack in Section 4. We now turn our attention to the link between the basic singularities

$$r^{1/2}\sin(\varphi/2) \text{ and } \rho_s^{-3/2}(\sin\theta_s)^{1/2}\sin(\varphi/2)$$

in expansions (2.2) and (2.9): one is obtained from the other by a Kelvin transformation (inversion and multiplication by ρ_s^{-1}). The lower singularity in expansion (2.2) and the basic singularity in the weighting function $\zeta^3(s; \cdot)$, which is intended for determining the value of the coefficient k_1 at the point s, are related in exactly the same way and, in fact,

$$r^{3/2}\sin(3\varphi/2) \text{ and } \rho_s^{-5/2}(\sin\theta_s)^{3/2}\sin(3\varphi/2) \tag{2.16}$$

The equality

$$\frac{1}{3}\frac{\partial}{\partial n}r^{3/2}\sin\frac{3\varphi}{2} = \frac{1}{2}r^{1/2}\sin\frac{\varphi}{2}$$

holds for the factors K_1 and k_1 in relation (2.2) (differentiation in the plane of the crack along a direction perpendicular to the front) and, according to the results obtained earlier[32,33] (see also Ref. 23, Ch. 12), the expression

$$-2\frac{\partial}{\partial n}\frac{1}{(2\pi)^{1/2}}r^{-1/2}H(s)\sin\frac{\varphi}{2} = \frac{1}{(2\pi)^{1/2}}r^{-3/2}H(s)\sin\frac{3\varphi}{2}$$

therefore serves as the principal singularity of the weighting function $V^3(H;\cdot)$, which enables us to find the weighted mean of the coefficient k_1 on the contour Γ (compare with the first formula of (2.11)).

We emphasize that differentiation across the crack front does not lead[34] to the correct answer (2.16) in the case of the weighting function $\zeta^3(s;\cdot)$. The point is that the derivative

$$-2\left(\frac{1}{2\pi^3}\right)^{1/2}\frac{\partial}{\partial n}\rho_s^{-3/2}(\sin\theta_s)^{1/2}\sin\frac{\varphi}{2}$$

becomes equal to $O(\rho_t^{-2}r^{-1/2})$, i.e., it also acquires a singularity on departure from the point s. As a result, the integral formula, similar to (2.11), with a weighting function obtained in such a way contains a linear combination of the quantities $k_1(s)$, $K_1(s)$ and $\partial_s K_1(s)$ on the left-hand side (for details, see Ref. 23, § 12.2).

Finally, we note that what has been said above only refers to the main singularities, but the regular parts and the “tails” of the asymptotic expansions, which depend, in particular, on the curvature κ, are not subject to the above relations. However, in the case of a circular crack, the operator $x \cdot \nabla x$, which preserves the harmonicity of the functions and the homogeneous boundary conditions (see Section 4) can be used to construct higher-order weighting functions.

3. Solutions of the variational inequality

The following equality holds

$$(bH, H)_\Gamma - (B(H), H)_\Gamma = \int_\Gamma b(s)H(s)^2 ds + \frac{1}{2}\int_\Gamma\int_\Gamma (H(t) - H(s))^2 Z(t,z)dtds \tag{3.1}$$

on account of the symmetry and positiveness of the kernel Z and the order of its singularity, which is indicated by formula (2.6), and, when $b > 0$, expression (3.1) is a quadratic norm in the Sobolev-Slobodetskii space $H_2^{1/2}(\Gamma)$. Henceforth, s and p in the notation $H_{p+}^s(\Gamma)$ indicate the smoothness and summability indices, and the plus sign indicates the convex set of non-negative functions. The named property of the form (3.1) enables us to change, using closure in inequality (1.3), to the arbitrary reference functions $X \in H_{2+}^{1/2}(\Gamma)$ and, thereby, to make it variational, that is, “generated functional of the energy” (3.1). The following assertion, which is proved in Refs. 4 and 14 (also, see Refs. 23, §12.8 and 35), establishes the existence and uniqueness of the solution of problem (1.3) as well as its smoothness.

Proposition 1. *When the condition*

$$b(s) > 0, \quad s \in \Gamma \tag{3.2}$$

is satisfied, the variational inequality (1.3) with an arbitrary right-hand side $F \in L_2(\Gamma)$ has a unique solution $H \in H_{2+}^{1/2}(\Gamma)$ and the estimate $||H; H_2^{1/2}(\Gamma)|| \le c||(F)_+; L_2(\Gamma)||$, in which c is a constant independent of F and $(t)_+ = (t + |t|)/2$ is the positive part of a number $t \in \mathbb{R}$, is true if, additionally, $F \in L_p(\Gamma)$ with a certain $p \in [2, +\infty)$ then $H \in H_p^1(\Gamma)$ *holds* $||H; H_p^1(\Gamma)|| \le c_p||(F)_+; L_2(\Gamma)||$.

Thus for the occurrence of a family of solutions, it is necessary that condition (3.2) should be violated,. We shall assume that

$$b(s) = b_\perp(s) - b_0, \quad b_0 > 0, \quad \int_\Gamma b_\perp(s)ds = 0 \tag{3.3}$$

Since the component $b_\perp$ has a zero mean, it is non-positive in a segment of the contour Γ, and this means that $b < 0$ in this segment.

We now turn our attention to the following fact: the difference between the two positive solutions H_1 and H_2 of problem (1.3) satisfies the homogeneous equation (1.7), that is, b_0 is an eigenvalue of the operator $b_\perp - B$. The main symbol of the integral operator (2.5) is equal to $-|\xi|/2$ (see Ref. 5 and, also, Ref. 23,§12.8, etc.). Hence in the case of the operator $b_\perp - B$, which is considered as unbounded in $L_2(\Gamma)$, there is a sequence of normal eigenvalues

$$\Lambda_0 \le \Lambda_1 \le \Lambda_2 \le \ldots \le \Lambda_n \le \ldots \to +\infty \tag{3.4}$$

We will assume that the constant term b_0 from formula (3.3) is identical to one of the eigenvalues in the sequence (3.4) and denote its multiplicity and the corresponding eigenfunctions, which have been orthonormalized in $L_2(\Gamma)$ and are smooth, by J and $H^{(k+1)}, \ldots, H^{(k+J)}$. If the right-hand side of F satisfies the orthogonality conditions

$$\int_\Gamma H^{(k+j)}(s)F(s)ds = 0, \quad j = 1, \ldots, J \tag{3.5}$$

then Eq. (1.7) has the particular solution H_0. The main assumption in the subsequent general constructions is concerned with the possibility of finding a *positive particular* solution

$$H_0(s) > 0, \quad s \in \Gamma \tag{3.6}$$

It is easy to show that a *non-negative* solution of Eq. (1.7) satisfies the variational inequality (1.3) and that, in the case of condition (3.6), at least small variations of the coefficients c_j preserve the required property in the case of the following general solution

$$H_0 + c_1 H^{(k+1)} + \ldots + c_j H^{(k+J)} \tag{3.7}$$

Proposition 2. *If b_0 is an eigenvalue from the sequence (3.4), then, when the requirements (3.3), (3.5) and (3.6) are satisfied, the variational inequality (1.3) admits of infinitely many solutions among the linear combinations (3.7). The sufficient condition, which ensures the non-negativity of the function (3.7), appears as: $m_1|c_1| + \ldots + m_J|c_J| \le m_0$. Here $m_0 = \min H_0(s) > 0$ and $m_j = \max|H^{(j)}(s)| > 0$, and the extrema are calculated over all points $s \in \Gamma$.*

An unlimited change in the coefficients c_j, while preserving the non-negativity of expression (3.7), is only possible in the case of a sign-definite eigenfunction $H^{(j)}$. Since the eigenfunctions are mutually orthogonal in $L_2(\Gamma)$, there is no more than one such eigenfunction. If $b_\perp = 0$ in representation (3.3), then on account of the existence of the difference in the integrand (2.5), the eigenfunction in question is constant.

It is possible to satisfy the requirements formulated in proposition (3.2) by a direct choice of the data: for $b_\perp = 0$ and any eigenvalue b_0 of the operator B, we calculate the right-hand side of F using formula (1.7) with any smooth positive function $H = H_0$ (the orthogonality conditions (3.5) are automatically satisfied). The main question, which is discussed in the following section, concerns the determination of the load (1.1) which ensures that appropriate data occur in problem (1.3).

4. A circular crack in the case of symmetrical loading

By virtue of the Papkovich–Neuber representation and the Kelvin transformation, the function $\zeta(r, \cdot)$ (see (2.9) and (2.10)) for a circular crack

$$M = \{x = (y, z) : |y| < R, z = 0\} \tag{4.1}$$

can be calculated explicitly: if $t = 0$ and the singularity is located at a point $(R, 0, 0) \in \Gamma$, then the trace on the upper edge M^+ has the form

$$\zeta(0; y, +0) = (4\pi^3 R)^{-1/2}(|y_1 - R|^2 + y_2^2)^{-1}(R^2 - y_1^2 - y_2^2)^{1/2} \tag{4.2}$$

In view of the axial symmetry, it is easy to recalculate $\zeta(t; \cdot)$ for an arbitrary point t and, using the second formula of (2.11), to calculate $K_1(t)$ for any load $g(y)$ (it is surprising that this simple representation of the SIF is not in the handbook, Ref. 36). If a symmetrical load $\pm g(|y|)$ is applied to the sides of the crack (4.1), it is more convenient to use another weighting function $V(1; \cdot)$ (see (2.8)) for which the trace on the side M^+ is as follows:

$$V(1; y, +0) = \pi^{-1/2} R^{1/2} (R^2 - |y|^2)^{-1/2} \tag{4.3}$$

By virtue of the integral representations (2.11), the SIF K_1, which is constant in the front Γ, is found using the formula (compare with Refs. 27 and 7)

$$K_1 = \frac{2}{(\pi R)^{1/2}} \int_0^R g(\mathbf{r})(R^2 - \mathbf{r}^2)^{-1/2} \mathbf{r} d\mathbf{r} \tag{4.4}$$

To calculate the "lowest" SIF $k_1(s) = k_1$, we can construct a weighting function $\zeta^3(t; \cdot)$. However, it is simpler to use the alternative technique of differentiation along the crack (both techniques were discussed at the end of Section 2). Note that the function $(x \cdot \nabla_x)v$ retains harmonicity and the following relations are satisfied

$$\partial_z(x \cdot \nabla_x)v(y, 0) = -\alpha(\mathbf{r}\partial_{\mathbf{r}} g(\mathbf{r}) + g(\mathbf{r})), \quad y \in M, \quad (x \cdot \nabla_x)v(y, 0) = 0, \quad y \in \mathbb{R}^2 \backslash M \tag{4.5}$$

where $\mathbf{r} = |y|$ and, correspondingly, $\mathbf{r}\partial_{\mathbf{r}} = y \cdot \nabla_y$. Since $\mathbf{r} = R - r$, the equality $\mathbf{r}\partial_{\mathbf{r}} = -R\partial_r + r\partial_{\mathbf{r}}$ holds in the crack M and, so, due to expansion (2.2), we have

$$(x \cdot \nabla_x)v(y, 0) = -\alpha(2\pi)^{-1/2} R\{K_1 r^{-1/2} - k_1 r^{1/2} + O(r)\}, \quad y \in M \tag{4.6}$$

Consequently, $(x \cdot \nabla_x)_v$ is the non-energy solution of Laplace equation with boundary conditions (4.5) but, by virtue of equality (4.3), the combination $(x \cdot \nabla_x)v(x) + \alpha K_1 R V(1; x)$, in which the singularity $O(r^{-1/2})$ has been removed, serves as a solution with a finite Dirichlet integral, and the SIF will be

$$\begin{aligned} \frac{R}{2}k_1 + \frac{1}{8}K_1 &= \frac{2}{(\pi R)^{1/2}} \int_0^R \{\mathbf{r}\partial_{\mathbf{r}} g(\mathbf{r}) + g(\mathbf{r})\}(R^2 - \mathbf{r}^2)^{-1/2} \mathbf{r} d\mathbf{r} = \\ &= \frac{2}{(\pi R)^{1/2}} \int_0^R \partial_{\mathbf{r}} g(\mathbf{r})(R^2 - \mathbf{r}^2)^{-1/2} \mathbf{r}^2 d\mathbf{r} + K_1 \end{aligned} \tag{4.7}$$

We emphasize that the second term on the left has arisen due to the addition of the weighting function.

If $g = 0$ on the edge Γ, then, when relation (4.4) is taken into account, equality (4.7) is converted by integration by parts to the form

$$k_1 = -\frac{1}{2}(\pi R^3)^{-1/2} \int_0^R g(\mathbf{r})(9R^2 - \mathbf{r}^2)(R^2 - \mathbf{r}^2)^{-3/2} \mathbf{r} d\mathbf{r} \tag{4.8}$$

From formula (4.8), it is easy to extract the trace of the weighting function $V^3(\cdot; 1)$ in M^+. On account of its strong singularity $O(r^{-3/2})$, the integral representation (4.8) is only correct in the case of a load which vanishes on the crack front.

We will now consider a specific integral operator B. Comparing expansions (2.10) and (4.2), we see that

$$Z(0, s) = \frac{1}{2\pi}\left[2R\sin\frac{s}{2R}\right]^{-2}$$

so, by formulae (2.6) and (2.5), this means that $Z^\# = 0$ and

$$B(H; s) = \frac{1}{2\pi} \int_\Gamma (H(t) - H(s))\left[2R\sin\frac{t - s}{2R}\right]^{-2} dt \tag{4.9}$$

It is clear that $\Lambda_0 = 0$ is the first and simple characteristic number of the operator (4.9) and that the corresponding eigenfunction $H^{(0)} = (2\pi R)^{-1/2}$ is constant. In order to find the other characteristic numbers $-\Lambda_m$ and eigenfunctions $H^{(m)}$, we substitute $H(s) = e^{\pm ims/R}$ into the quadratic form (3.1) with $b = -\Lambda_m$ and, taking equality (4.9) into account, after changing to the variable $\sigma = R^{-1}s$, we obtain

$$0 = -\Lambda_m R \int_0^{2\pi} |e^{\pm im\sigma}|^2 d\sigma + \frac{1}{16\pi} \int_0^{2\pi}\int_0^{2\pi} |e^{\pm im\sigma} - e^{\pm im\tau}|^2 \left[\sin\frac{\tau - \sigma}{2}\right]^{-2} d\sigma d\tau =$$

$$= -2\pi R\Lambda_m + \frac{1}{4\pi}\int_0^{2\pi} \frac{|e^{\pm im\tau}|^2}{|e^{\pm i(\sigma-\tau)/2}|^2}\int_0^{2\pi} \frac{|e^{\pm im(\sigma-\tau)}-1|^2}{|e^{\pm i(\sigma-\tau)}-1|^2} d\sigma d\tau =$$

$$= -2\pi R\Lambda_m + \frac{1}{2}\int_0^{2\pi}\left|\sum_{n=0}^{m-1} e^{\pm in\zeta}\right|^2 d\zeta = -2\pi R\Lambda_m + \pi m$$

Hence, in the sequence (3.4), $\Lambda_m = (2R)^{-1}m$, where $m \in \{0, 1, 2, \ldots\}$ and the two eigenfunctions

$$H^{(m1)}(s) = (\pi R)^{-1/2}\sin(mR^{-1}s), \quad H^{(m2)}(s) = (\pi R)^{-1/2}\cos(mR^{-1}s)$$

correspond to each positive characteristic number. Since the sines and cosines constitute the basis in the space $L_2(\Gamma)$, there are no other characteristic numbers.

Everything is now ready to formulate a corollary of Proposition 2 for a circular crack of radius R.

Proposition 3. *Suppose the relations*

$$F(\tau; s) = \tau F_0, \quad F_0 < 0, \quad b = -(2R)^{-1}m, \quad m \in \{1, 2, \ldots\} \tag{4.10}$$

hold in the case of the data for the variational inequality (1.3) with the operator (4.9) in a circle $[y : \mathbf{r} = R]$. *Then, for any constants* c_1 *and* c_2 *which satisfy the requirement* $(c_1^2 + c_2^2)^{1/2} \le -F_0$, *the function*

$$\Gamma \ni s \mapsto H(\tau; s) = -2\tau Rm^{-1}\{F_0 + c_1\sin(mR^{-1}s) + c_2\cos(mR^{-1}s)\} \tag{4.11}$$

is non-negative and satisfies Eq. (1.7), *and so it is a solution of the variational inequality (1.3).*

Since $K_1(s) = K_{1c}$ on the developing crack front (4.1), the equality $H = h$ follows from definition (2.12). If $c_1 = c_2 = 0$ on the right-hand side of the equality (4.11), then h is constant, that is, the crack remains circular.

Suppose $m = 1$ and $|c_1| + |c_2| = 0$. By choosing the origin on the circle Γ in an appropriate manner, we obtain that relations $c_1 = 0$ and $c_2 < 0$ hold. Now,

$$h(s) = -2\tau R\{F_0 + c_2\cos(R^{-1}s)\} \tag{4.12}$$

but simple calculations show that, in spite of the lack of axial symmetry with respect to the centre of the crack $M = M(0)$, the front $\Gamma(\tau)$, which is found using formulae (1.2) and (4.2), remains circular. Here, the error $O(\tau^2)$ does not exceed the accuracy of the variational-asymptotic model and the crack $M(\tau)$ differs only slightly from a circle of radius $R(1 + 2\tau|F_0|)$ and centre displaced relative to its initial position by a distance of $2\tau R|c_2|$.

If, however, $m \ge 2$ and, as previously, $|c_1| + |c_2| \ne 0$, then the crack $M(\tau)$ loses its axial symmetry due to the formation of m "lobes". We emphasize that the condition $h(s) \ge 0$ for the irreversibility of the fracture process is satisfied and that distortion of the shape only occurs for the reason that there is a difference in the rate of advance of points of the front on its different segments.

In the case when $K_1(s) = K_{1c}$, the data (2.13) and (2.15) of the variational inequality (1.3) are the same for both criteria, the force criterion and the energy criterion. Propositions (4.10) are realized under the conditions

$$k_1 = \frac{K_{1c}}{R}\left(\frac{3}{4} + m\right), \quad m \in \{1, 2, \ldots\} \tag{4.13}$$

$$F_0 = K_{1c}^{-1}\frac{2}{(\pi R)^{1/2}}\int_0^R g'(\mathbf{r})(R^2 - \mathbf{r}^2)^{-1/2}\mathbf{r}d\mathbf{r} < 0 \tag{4.14}$$

If the forces (1.1) which open the crack are constant, then, by virtue of (4.7) $k_1(s) = 7K_{1c}(4R)^{-1}$, and requirement (4.13) is satisfied when $m = 1$. Inequality (4.14) holds when $g' < 0$. Consequently, if, as the circular crack grows, the forces are attenuated so that the crack remains an equilibrium but critical crack, bifurcations of its shape are possible. However, they only consist of a change in the position of the crack and of fluctuations of its centre.

Suppose it is possible in some way to succeed in making the loading on the sides of the crack variable, for example, by adding constricting forces close to the centre $x=0$. In the case of a monotonically increasing function $\mathbf{r} \mapsto g(\mathbf{r})$, the first term on the right-hand side of (4.7) becomes positive. It is easy to show that, by the choice of the function g, condition (4.3) can be successfully satisfied for any specified $m=2, 3, \ldots$. In this case, bifurcation of the crack front, which accompanies the formation of m lobes, is also possible in the case of a decreasing load.

We will now describe a hypothetical scenario for a quasistatic fracture process. The development of a circular crack is unstable and, at any instant, the quasistatic fracture process can break off into a dynamic process, which is not described by the proposed model (an avalanche-type of crack growth). When $m \geq 2$, the bifurcation of the front removes the axial symmetry and, after passing to a perturbed, non-circular contour in the case of the new variational inequality, which is found using the scheme described in Section 1, the fracture process is localized in the concave zones of the front. Here, it is necessary to correlate the bifurcations with spontaneous interruptions in the monotonio growth of the crack, since it is not observed until the front becomes circular once again (with regard to this last point, see Ref. 7).

An analysis of the integral representations (4.4) and (4.7) for the SIF shows that, when relation (4.13) is satisfied for large m, the load is concentrated close to the edge of the crack. We also draw attention to the fact that, according to equality (4.8) in the case of a load g which vanishes on the edge Γ, the coefficient k_1 is negative, that is $b>0$ by virtue of definition (2.13) and, according to Proposition 1, the solution of the variational inequality is found to be unique, that is, there are no bifurcations.

Acknowledgement

This research was supported financially by the Russian Foundation For Basic Research (03-01-00835).

References

1. Morozov EM. The variational principle in fracture mechanics. *Dokl Akad Nauk SSSR* 1969;**184**(6):1308–11.
2. Nemat-Nasser, Sumi Y, Keer LM. Unstable growth of tension cracks in brittle solids: stable and unstable bifurcations, snap-through and imperfection sensitivity. *Intern J Solids Structures* 1980;**16**:1017–33.
3. Nguen Quoc Son. Stabilite et bifurcation en rupture et en plasticite. *Cr Acad Sci Paris Sér II* 1981;**292**:817–20.
4. Nazarov SA. Derivation of a variational inequality for the form of a small increment in a tensile crack. *Izv Akad Nauk SSSR MTT* 1989;**2**:152–60.
5. Nazarov SA, Polyakova OR. The equivalence of fracture criteria for a tensile crack in an elastic space. *Izv Ross Akad Nauk MTT* 1992;**2**:101–13.
6. Kolton LH, Nazarov SA. Quasistatic propagation of a mode-I crack in an elastic space. *Cr Acad Sci Paris Ser II* 1992;**315**:1453–7.
7. Bach M, Nazarov SA, Wendland WL. Propagation of a penny shaped crack under the Irwin criterion. In: Bach M, et al., editors. *Analysis, Numerics and Applications of Differential and Integral Equations: Pitman Research Notes in Mathematics*, vol. 379. Harlow, UK: Addison Wesley Longman Ltd.; 1998. p. 17–21.
8. Bach M, Nazarov SA, Wendland WL. Stable propagation of a mode-1 crack in an isotropic elastic space. Comparison of the Irwin and the Griffith approaches. In: Ricci PE, editor. *Problemi Attuali dell'Analisi e della Fisica Matematica*. Roma: MM, Aracne Edit; 2000. p. 167–89.
9. Nazarov SA. Interaction of cracks accompanying brittle fracture. Force and energy approaches. *Prikl Mat Mekh* 2000;**64**(3):484–96.
10. Zakharevich IS. The variation of the solutions of integrodifferential equations of mixed problems in the theory of elasticity accompanying a variation of the domain. *Prikl Mat Mekh* 1985;**49**(6):961–8.
11. Gao H, Rice JR. Somewhat circular tensile cracks. *Intern J Fracture* 1987;**33**:155–74.
12. Rice JR. First-order variation in elastic fields due to variation in location of a planar crack front. *Trans ASME J Appl Mech* 1985;**52**:571–9.
13. Leblond J-B, Lazarus VS-E, Mouchrif S. Crack paths in three-dimensional elastic solids. II. Three-term expansion of the stress intensity factors – applications and perspectives. *Intern J Solids Structures* 1999;**36**:105–42.
14. Kolton LG, Nazarov SA. Variation of the form of the edge of a plane locally non-equilibrium normal tensile crack. *Izv Ross Akad Nauk MTT* 1997;**3**:125–33.
15. Sih GC, Liebouwitz H. Mathematical theories of britte fracture. In: Liebowitz H, editor. *Fracture*, vol. 2. N.Y.; L: Acad. Press; 1968. p. 67–190.
16. D'Ecatha, Labbens R. A remark on two brittle fracture criteria in mode 1 three-dimensional problems. *J Mec Appl* 1978;**2**:524–52.
17. Ohtsuka K. Mathematical analysis of 3-d fracture phenomenon by Griffith's energy balance theory under increasing loads. *Theoret and Appl Mech* 1996;**45**:99–103.
18. Leblond J-B, Mouchrif S-E, Perrin G. The tensile tunnel-crack with a slightly wavy front. *Intern J Solids Structures* 1996;**33**:1995–2022.
19. Kolton LG. Slow growth of a system of cracks. *Izv Akad Nauk SSSR MTT* 1989;**5**:95–100.
20. Arutyunyan NKh, Movchan AB, Nazarov SA. The behaviour of the solutions of problems in the theory of elasticity in unbounded domains with paraboloidal and cylindrical inclusions and cavities. *Uspekhi Mekhaniki* 1987;**4**:3–91.
21. Nazarov SA. Asymptotics of the solution to the Neumann problem in a domain with singular point of peak exterior type. *Russ J Math Phys* 1996;**4**(2):217–50.
22. Rabotnov YuN. *The Mechanics of Deformable Solid*. Moscow: Nauka; 1979.

23. Nazarov SA, Plamenevsky BA. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. Berlin; New York: Walter de Gruyter; 1994. p. 525.
24. Nazarov SA. The Local stability and instability of normal tensile cracks. *Izv Akad Nauk SSSR MTT* 1988;**3**:124–9.
25. Bueckner HF. A novel principle for the computation of stress intensity factor. *ZAMM* 1976;**50**:529–46.
26. Maz'ya VG, Plamenevskii BA. The coefficients in the asymptotics of the solutions of elliptic boundary-value problems in domains with conical points. *Math Nachr* 1977;**76**:29–60.
27. Bueckner HF. Weight functions and fundamental fields for the penny-shaped and the half-plane crack in three-dimensional space. *Inter J Solids Structures* 1987;**23**(1):57–93.
28. Maz'ya VG, Plamenevskii BA. The coefficients in the asymptotics of solutions of elliptic boundary-value problems close to an edge. *Dokl Akad Nauk SSSR* 1976;**229**(1):33–6.
29. Maz'ya VG, Rossmann J. Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten. *Math Nachr* 1988;**138**:27–53.
30. Rice JR. Weight function theory for three-dimensional elastic crack analysis. Fracture Mechanics: Perspectives and Directions: 20th Symp. ASTM STP 1020, Philadelphia, 1989. p. 29.
31. Nazarov SA, Plamenevskii BA. Self-adjoint problems with emission conditions on the edges of a boundary. *Algebra i Analiz* 1992;**4**(3):196–225.
32. Nazarov SA. Weight functions and invariant integrals. *Vychislitel'naya Mekhanika Deformiruyemogo Tverdogo Tela* 1990;**1**:17–31.
33. Nazarov SA. A crack in the joint of anisotropic bodies. Stress singularities and invariant integrals. *Prikl Mat Mekh* 1998;**68**(3):489–502.
34. Rossmann J, Sändig A-M. Formulas for the coefficients in the asymptotics of solutions of boundary value problems for order systems near edges. *ZAMM* 1996;**76**(4):181–4.
35. Bach M, Nazarov SA. Smoothness properties of solutions to variational inequalities describing propagation of mode-1 cracks. *Mathematical Aspects of Boundary Element Method.* Palaiseau, 1998. London: CRC: Chapman and Hall; 2000, p. 23–32.
36. Stress Intensity Factors Handbook. Oxford, etc., 1987.

Translated by E.L.S.